

# D-Branes, Holonomy and M-Theory

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We show that M-theory on spaces with irreducible holonomy represent Type IIA backgrounds in which a collection of D6-branes wrap a supersymmetric cycle in a manifold with a holonomy group different from the one appearing in the M-theory description. For example, we show that D6-branes wrapping a supersymmetric four-cycle on a manifold with  $G_2$  holonomy is described in eleven dimensions by M-theory on a space with Spin(7) holonomy. Examples of such Type IIA backgrounds which lift to M-theory on spaces with  $SU(3)$ ,  $G_2$ ,  $SU(4)$  and Spin(7) holonomy are considered. The M-theory geometry can then be used to compute exact quantities of the gauge theory on the corresponding D-brane configuration.

March 2001

## 1. Introduction

One of the important lessons of the duality era has been that string theory backgrounds can be naturally embedded in an eleven dimensional theory. The simplest example of this embedding is the uplift of any background of Type IIA string theory to M-theory [1,2]. In particular, any solution of the equations of motion described by the ten dimensional metric  $g_{\mu\nu}$ , the Ramond-Ramond one form  $C_\mu$  and the dilaton  $\phi$  uplifts in eleven dimensions to a solution of eleven dimensional supergravity with the following eleven dimensional metric

$$ds_{11}^2 = e^{-\frac{2\phi}{3}} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4\phi}{3}} (dx_{11} + C_\mu dx^\mu)^2. \quad (1.1)$$

The uplift of the rest of the massless bosonic fields of Type IIA string theory corresponds to turning on the three-form gauge field of eleven dimensional supergravity. Therefore, backgrounds of Type IIA string theory which only source the fields  $g_{\mu\nu}$ ,  $C_\mu$  and  $\phi$  must be described in eleven dimensions by a purely gravitational background without any flux.

On the other hand, supersymmetric M-theory vacua which are purely gravitational are completely classified. Supersymmetric compactifications of eleven dimensional supergravity to  $d + 1$ -dimensional Minkowski space  $\mathbf{R}^{1,d}$  appear whenever the  $D \equiv 10 - d$  dimensional compactification manifold  $\mathbf{X}$  admits covariantly constant spinors<sup>1</sup>. The only solution in this class with maximal supersymmetry is when  $\mathbf{X}$  is a torus  $\mathbf{T}^D$ . Solutions with reduced supersymmetry can be obtained when the compactification space  $\mathbf{X}$  has covariantly constant spinors which do not span the full  $SO(D)$  spinor. Such  $D$ -dimensional spaces have a holonomy group  $G$  such that at least one component of the  $SO(D)$  spinor is left invariant by the action of  $G$ . The list [6] of irreducible holonomy groups preserving some supersymmetry and the corresponding number of unbroken supercharges in  $\mathbf{R}^{1,d}$  is :

- $D = 4$ ,  $G = SU(2)$ , 7 dim.  $\mathcal{N} = 1$  SUSY, 16 real supercharges
- $D = 6$ ,  $G = SU(3)$ , 5 dim.  $\mathcal{N} = 1$  SUSY, 8 real supercharges
- $D = 7$ ,  $G = G_2$ , 4 dim.  $\mathcal{N} = 1$  SUSY, 4 real supercharges
- $D = 8$ ,  $G = SU(4)$ , 3 dim.  $\mathcal{N} = 2$  SUSY, 4 real supercharges
- $D = 8$ ,  $G = \text{Spin}(7)$ , 3 dim.  $\mathcal{N} = 1$  SUSY, 2 real supercharges

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<sup>1</sup> There are many other interesting supersymmetric solutions if one allows for non-trivial background fluxes. For example, one can have AdS vacua (see [3] for a nice review of some such solutions) and warped compactifications (see e.g. [4,5]).

The number of unbroken supercharges is given by the number of singlets in the decomposition of the spinor of  $SO(D)$  under the reduced holonomy group  $G \subset \text{Spin}(D)$ . Manifolds with  $SU(2)$ ,  $SU(3)$  and  $SU(4)$  holonomy are Calabi-Yau spaces while manifolds with  $G_2$  and  $\text{Spin}(7)$  holonomy are real Ricci flat manifolds<sup>2</sup>.

In this paper we show that M-theory compactified<sup>3</sup> on certain spaces with reduced holonomy appear as the local eleven dimensional description of Type IIA D6-branes wrapping supersymmetric cycles on certain lower dimensional spaces with a different holonomy group than the one in the M-theory description. It is crucial that we use D6-branes in the Type IIA construction since this is the only<sup>4</sup> brane which lifts in eleven dimensions to pure geometry. Supersymmetry severely constraints the types of lifts that are possible. The basic geometry that appears in eleven dimensions near the location of the D6-branes is that of an ALE fibration over the supersymmetric cycle in which we wrapped the D6-branes. This fibration has a different holonomy group than that of the manifold we started with in Type IIA. We consider the possibility of adding curved orientifold six planes (O6-planes) by modding out by worldsheet parity combined with an appropriate involution on the geometry. Then, the type of ALE fibration appearing in eleven dimensions depends on whether there is an orientifold plane in the Type IIA description. The gauge theory on the D6-branes appears in the M-theory description from singularities in the geometry. In the various examples we consider, we notice that the M-theory geometrical description can be used to compute quantities – such as prepotentials or superpotentials – of the gauge theory living on the D6-branes.

Our results extend the work of [8][9] where a configuration of D6 branes wrapping the  $\mathbf{S}^3$  which appears in the deformation of the conifold singularity of a Calabi-Yau three-fold was represented in M-theory as compactification on a certain space with  $G_2$  holonomy. For example, we show that M-theory on an orbifold of the spin bundle over  $\mathbf{S}^4$   $S(\mathbf{S}^4)$  – which admits a metric with  $\text{Spin}(7)$  holonomy – appears as the strong coupling description of D6-branes wrapping the supersymmetric four-cycle of the bundle of anti-self-dual

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<sup>2</sup> Eight dimensional hyper-Kähler manifolds with  $Sp(2)$  holonomy will not be considered in this paper. They give rise to three dimensional theories with  $\mathcal{N} = 3$  supersymmetry.

<sup>3</sup> We will consider non-compact spaces so strictly speaking we are not compactifying. Therefore, the constraint [7] imposed by tadpole cancellation is lifted since the flux lines can escape to infinity.

<sup>4</sup> The magnetic dual object, the D0-brane, also lifts to pure geometry.

two-forms over  $\mathbf{S}^4 \times \Sigma(\mathbf{S}^4)$  which admits a  $G_2$  holonomy metric. This type of correspondence is extended to all other cases. Namely, we show that M-theory on certain singular spaces with  $SU(3)$ ,  $G_2$ ,  $SU(4)$  and  $\text{Spin}(7)$  holonomy describe in eleven dimensions a configuration of D6-branes wrapped on a supersymmetric cycle on a manifold with different holonomy group. For example, we show that M-theory on the Calabi-Yau three-folds used in [10] to geometrically engineer supersymmetric gauge theories appear as the local description of D6-branes wrapping two-cycles in manifolds with  $SU(2)$  holonomy. We would like to emphasize that the M-theory descriptions that we propose in this paper are appropriate descriptions of the Type IIA brane configurations near the location of the D6-branes. Globally, the eleven dimensional geometry is more complicated. In this paper we will provide simple examples of the local description of the wrapped supersymmetric D6-branes configurations in terms of eleven dimensional geometry.

This phenomena, apart from giving a physical rational for the existence of manifolds with reduced holonomy, can be useful in deriving string theory dualities using geometrical transitions. Recently, Atiyah, Maldacena and Vafa [9] have lifted the description of Type IIA backgrounds on two different Calabi-Yau three-folds to eleven dimensions and have found that the two different Type IIA backgrounds are described in M-theory by two different  $G_2$  holonomy spaces which are related to each other by a flop transition [9]. Therefore, by going through a smooth physical transition in M-theory, like a flop in a  $G_2$  holonomy manifold, they derive the physical equivalence of string theory in the two different Calabi-Yau manifolds, which was previously conjectured by Vafa [11]. Various aspects of this duality and generalizations to other three-folds have recently appeared in [12,13,14,15]. The lifts proposed in this paper provide quantitative information about the gauge theory on the branes and can be useful in deriving possible new dualities among Type IIA vacua by going through a geometrical transition in the M-theory geometries we discuss.

The rest of the paper is organized as follows. Section 2 briefly reviews the eleven dimensional description of Type IIA D6-branes in flat space with and without an orientifold six-plane. In section 3 we discuss the gauge theories one gets by wrapping D6-branes on supersymmetric cycles. We also exhibit the general features of the eleven dimensional geometry that represents the various wrapped D6-brane configurations. Sections 4-7 give specific examples of lifts of Type IIA backgrounds with branes on certain manifolds with one holonomy group in terms of M-theory on spaces with a different holonomy group.

## 2. The Geometry of D6-branes and the O6-Plane

Any solution of the Type IIA string equations of motion only sourcing the Type IIA fields  $g_{\mu\nu}$ ,  $C_\mu$  and  $\phi$  is described in eleven dimensions by a purely gravitational background. A simple example of this phenomena is provided by a collection of D6-branes. The eleven dimensional description of  $N$  separated D6-branes can be obtained by using (1.1) on the Type IIA supergravity solution of [16]. It is given by

$$ds_{11}^2 = -dx_0^2 + dx_1^2 + \dots + dx_6^2 + ds_{TN}^2, \quad (2.1)$$

where  $ds_{TN}^2$  is the metric of the Euclidean multi-centered Taub-NUT space [17]

$$\begin{aligned} ds_{TN}^2 &= H d\vec{r}^2 + H^{-1} (dx_{11} + C_\mu dx^\mu)^2 \\ \vec{\nabla} \times \vec{C} &= -\vec{\nabla} H \\ H &= 1 + \frac{1}{2} \sum_{i=1}^N \frac{g_s \sqrt{\alpha'}}{|\vec{r} - \vec{r}_i|}, \end{aligned} \quad (2.2)$$

which is a hyper-Kähler metric on a  $U(1)$  bundle over  $\mathbf{R}^3$ . The  $x^{11}$  coordinate is periodic and absence of conical singularities at  $\vec{r} = \vec{r}_i$  requires its periodicity to be  $2\pi g_s \sqrt{\alpha'}$ . The circle fiber is identified with the M-theory circle which has the expected radius  $R = g_s \sqrt{\alpha'}$  at infinity. Therefore, a collection of D6-branes in flat space can be represented in M-theory by a four dimensional manifold with  $SU(2)$  holonomy.

In this paper we take the D6-branes to be coincident – all  $\vec{r}_i = 0$  – which gives rise to the familiar enhanced  $SU(N)$  gauge symmetry. The M-theory description of this configuration near the location of the D6-branes at  $\vec{r} \simeq 0$  is given by

$$\begin{aligned} ds_{ALE}^2 &\simeq H d\vec{r}^2 + H^{-1} (dx_{11} + C_\mu dx^\mu)^2 \\ H &\simeq \frac{N g_s \sqrt{\alpha'}}{2r}, \end{aligned} \quad (2.3)$$

which is the metric on the  $A_{N-1}$  ALE space [18] near an orbifold singularity. Here the  $SU(N)$  gauge symmetry arises from membranes wrapping the collapsed two-cycles [2]. In the limit when all  $\vec{r}_i = 0$  the metric on the  $A_{N-1}$  ALE space [18] degenerates to the metric on the  $\mathbf{C}^2/\mathbf{Z}_N$  orbifold, where  $\mathbf{Z}_N$  acts by

$$\begin{aligned} z_1 &\rightarrow e^{\frac{2\pi i}{N}} z_1 \\ z_2 &\rightarrow e^{-\frac{2\pi i}{N}} z_2. \end{aligned} \quad (2.4)$$

Three of the coordinates which are acted on are the original transverse directions of the D6 branes. The fourth orbifolded direction corresponds to the M-theory circle whose asymptotic radius is now infinity. Therefore, Type IIA string theory with  $N$  coincident D6-branes is locally described in eleven dimensions by the  $A_{N-1}$  ALE singularity.

One can also find the M-theory realization of a collection of D6-branes sitting on top of an orientifold six-plane (O6-plane). The O6-plane appears as the fixed locus obtained by modding out string theory by the orientifold group  $G = \{1, \Omega(-1)^{F_L} R_7 R_8 R_9\}$ , where  $\Omega$  is worldsheet parity,  $F_L$  is the left-moving space-time fermion number and  $R_i$  is a reflection along the  $x^i$ -th coordinate. In this paper we consider M-theory lifts involving the O6-plane which carries -2 units of D6-brane charge<sup>5</sup> (the  $O6^-$  plane).  $N$  coincident D6-branes on top of this orientifold plane gives rise to enhanced  $SO(2N)$  gauge symmetry. It was shown in [19,20] that this O6-plane is represented in M-theory by the Atiyah-Hitchin space [21] and that the combined system of  $N$  D6-branes on top of the O6-plane is described locally by the  $D_N$  ALE singularity. This is the orbifold singularity  $\mathbf{C}^2/\hat{\mathbf{D}}_{N-2}$ , where  $\hat{\mathbf{D}}_{N-2}$  is the Dihedral group with a  $\mathbf{Z}_2$  central extension<sup>6</sup>. The  $SO(2N)$  gauge symmetry arises in the M-theory description from membranes wrapping the shrunken two-cycles at the orbifold singularity.

To summarize,  $N$  coincident D6-branes in flat space are described in eleven dimensions by an  $A_{N-1}$  ALE singularity and  $N$  D6-branes on top of an  $O6^-$ -plane by a  $D_N$  ALE singularity. The corresponding orbifold groups act on the normal direction of the D6-branes and on the M-theory circle. In the rest of the paper we will explore the M-theory description of D6-branes and D6-branes on top of an O6-plane wrapping non-trivial supersymmetric cycles in various non-trivial backgrounds.

### 3. General Strategy

We want to consider Type IIA supersymmetric configurations of D6-branes wrapping cycles in spaces with reduced holonomy and to find the corresponding M-theory description. In order for the D6-branes to preserve some of the supersymmetry left unbroken by the compactification space, the cycle which is wrapped must be supersymmetric [23,24]. The

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<sup>5</sup> In this paper all charges are measured in the quotient space.

<sup>6</sup> This discrete subgroup of  $SU(2)$  has order  $4N - 8$  and has two generators  $a$  and  $b$  which satisfy  $a^{2N-4} = b^4 = 1$  and  $ba = a^{-1}b$ . The action of  $\hat{\mathbf{D}}_{N-2}$  on  $\mathbf{C}^2$  can be found, for example, in Table 1 of [22].

BPS condition on a supersymmetric cycle on a manifold with holonomy group  $G$  can be written in terms of the corresponding  $G$ -structure. The following table summarizes the classification of supersymmetric cycles (table 3 in [25]):

p+1	$SU(2)$	$SU(3)$	$G_2$	$SU(4)$	$Spin(7)$
2	divisor/SLag	holomorphic	—	holomorphic	—
3	—	SLag	associative	—	—
4	$X$	divisor	coassociative	Cayley	Cayley
5	—	—	—	—	—
6	—	$X$	—	divisor	—
7	—	—	$X$	—	—
8	—	—	—	$X$	$X$

Table 1. Supersymmetric Cycles in irreducible holonomy manifolds.

All the above cycles preserve one-half of supersymmetry except the Cayley cycles of a Calabi-Yau four-fold which instead preserve one quarter of the supersymmetries. Therefore, the supersymmetric Type IIA backgrounds with wrapped D6-branes which can be described by M-theory on a manifold with irreducible holonomy are severely constrained by supersymmetry.

There are essentially two types of situations that need to be considered:

1) The D6-branes wrap a supersymmetric cycle in such a way that the D6-branes fill completely the space transverse to the IIA compactification manifold. The field theory one obtains lives in the transverse Minkowski space. Then the local M-theory description is given by a manifold one dimensional higher in the list of manifolds with irreducible holonomy. In this fashion one can find examples of the following lifts of holonomy groups (IIA  $\rightarrow$  M):  $SU(3) \rightarrow G_2$  and  $G_2 \rightarrow Spin(7)$ .

2) The D6-branes wrap a supersymmetric cycle in such a way that the D6-branes are codimension one in the transverse Minkowski space to the IIA compactification manifold. The field theory one obtains lives in codimension one on the transverse space. Then the local M-theory description – near the D6-branes – is given by a manifold of two

dimensions higher in the list of manifolds with irreducible holonomy. In this fashion one can find examples of the following lifts of holonomy groups (IIA  $\rightarrow$  M):  $SU(2) \rightarrow SU(3)$  and  $SU(3) \rightarrow SU(4)$ .

The effective gauge theory living on the D-branes is a topological field theory [24]. This topological field theory is a gauge theory in which the scalars parametrizing the positions of the D-branes are twisted and transform as sections of the normal bundle<sup>7</sup>. Therefore, given a supersymmetric cycle one can construct the supersymmetric gauge theory by analyzing the normal bundle of the supersymmetric cycle. In this paper we will concentrate on supersymmetric cycles which are *rigid* so that there are no scalars describing the possible deformations of the cycle inside the curved manifold. When the D6-branes are codimension one, we get a massless real scalar field which is part of the vector multiplet. It describes the positions of the D6-branes in the transverse direction. The M-theory realization of these backgrounds with D6-branes will geometrically engineer these gauge theories.

The local geometry of the wrapped  $N$  D6-branes in M-theory is as follows. Near the location of the D6-branes, the D6-branes are represented in M-theory by the  $A_{N-1}$  ALE singularity. Since some of the transverse directions of the D6-branes are curved and non-trivially fibered over the supersymmetric cycle, the lift to eleven dimensions is just given by the  $A_{N-1}$  ALE singularity fibered over the supersymmetric cycle. As we will show in the sections that follow we will be able to recognize such geometries as particular manifolds with irreducible holonomy.

We can also consider the situation in which we mod out the geometry on which the D6-branes are embedded by an involution  $\sigma$ . We will require that the involution have as fixed set the supersymmetric cycle on which the D-branes are wrapped and that the fixed set satisfies the corresponding supersymmetric cycle condition. In order to obtain a supersymmetric configuration this involution must be accompanied by an orientifold projection. The appropriate orientifold group is given<sup>8</sup> by  $G = \{1, \Omega(-1)^{F_L} \sigma\}$ . Therefore, by performing this orientifold projection, one obtains a curved orientifold six-plane (O6-plane) with  $N$  D-branes on top of it. Following the same argument we used to uplift the description of the curved D6-branes to M-theory we see that the M-theory description

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<sup>7</sup> Recently, supergravity duals of these topological field theories have been constructed, see e.g [26].

<sup>8</sup> In the situation where the D6-branes are codimension one in the transverse space, one must also reflect along the transverse coordinate.



of the curved D6+O6 configuration is given by the  $D_N$  ALE singularity fibered over the corresponding supersymmetric cycle. In the examples we will consider we will identify this fibration with a certain reduced holonomy manifold.

In the following sections we will provide concrete examples of these lifts.

#### 4. From $SU(2)$ holonomy to $SU(3)$ holonomy via D6-Branes

We lift various IIA configurations and recover the Calabi-Yau three-folds used, for example, in [10] to geometrically engineer four dimensional  $\mathcal{N} = 2$  gauge theories.

The simplest<sup>9</sup> case to consider is the M-theory description of  $N$  D6-branes wrapping the  $\mathbf{S}^2$  in  $T^*\mathbf{S}^2$ , the cotangent bundle of  $\mathbf{S}^2$ . This space appears when resolving or deforming an  $A_1$  singularity of K3. For instance,  $T^*\mathbf{S}^2$  is described by

$$z_1^2 + z_2^2 + z_3^2 = r. \quad (4.1)$$

If we rewrite  $z_j = a_j + ib_j$  for  $j = 1, 2, 3$  and take  $r$  to be real and positive, the real and imaginary parts of (4.1) lead to  $\vec{a}^2 - \vec{b}^2 = r$  and  $\vec{a} \cdot \vec{b} = 0$ . Since  $\vec{u} \cdot \vec{u} = 1$ , where  $\vec{u} = \vec{a}/\sqrt{\vec{b}^2 + r}$ ,  $\vec{u}$  generates an  $\mathbf{S}^2$ . Moreover, since  $\vec{b} \cdot \vec{u} = 0$ ,  $b_i$  spans the cotangent directions of  $\mathbf{S}^2$ , so (4.1) describes the cotangent bundle of  $\mathbf{S}^2$ . This space is topologically  $\mathbf{R}^2 \times \mathbf{S}^2$  and admits a hyper-Kähler metric with  $SU(2)$  holonomy. The  $\mathbf{S}^2$  is a holomorphic cycle and branes wrapping it preserve one-half of the supersymmetries left unbroken by the geometry.

We now consider  $N$  D6-branes wrapped on the supersymmetric  $\mathbf{S}^2$  inside  $\mathbf{R}^{1,5} \times T^*\mathbf{S}^2$  and stretched along the  $x^1 \dots x^4$  directions. Since the normal bundle of  $\mathbf{S}^2$   $N(\mathbf{S}^2)$  is trivial, the field theory living on the branes is a supersymmetric five-dimensional pure  $SU(N)$  gauge theory with eight real supercharges<sup>10</sup>. Alternatively, one can simply derive [28] this gauge theory by representing the wrapped D6-branes by  $N$  fractional [29] D4-branes at a  $\mathbf{C}^2/\mathbf{Z}_2$  orbifold singularity.

Now, in eleven dimensions we must obtain a geometrical background which leads to such a five-dimensional gauge theory. As explained in the previous section the geometry

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<sup>9</sup> Lifting to eleven dimensions a configuration of D6-branes wrapped on the entire K3 follows trivially from the discussion on the previous section. The corresponding eleven dimensional geometry is just  $\mathbf{R}^{1,2} \times \text{Taub-NUT} \times \text{K3}$ .

<sup>10</sup> The vector multiplet has a real scalar so there is a non-trivial Coulomb branch whose prepotential is constrained by gauge invariance to be cubic in the vector superfields [27].

we get is that of an  $A_{N-1}$  ALE singularity fibered over the  $\mathbf{S}^2$ . This geometry is indeed a Calabi-Yau three-fold [10] and gives rise to a five dimensional  $SU(N)$  gauge theory. The exact prepotential in the Coulomb branch of this gauge theory was computed from M-theory on this three-fold geometry in [30]. The computation reduces to evaluating the classical intersection numbers of the divisors one obtains when resolving the curve of  $A_{N-1}$  singularities which when blown up break the  $SU(N)$  gauge group to its maximal torus.

In the Type IIA description one can perform an orientifold projection combined with the following involution

$$\sigma : \begin{cases} z_i & \rightarrow & \bar{z}_i \\ x_5 & \rightarrow & -x_5 \end{cases} \quad i = 1, 2, 3. \quad (4.2)$$

This involution keeps fixed the real part of equation (4.1) fixed which describes the supersymmetric  $\mathbf{S}^2$ . Therefore, we obtain an O6-plane wrapping the  $\mathbf{S}^2$ . Then, the gauge theory on the D6-branes is a five-dimensional  $SO(2N)$  gauge theory with eight real supercharges.

It is simple to incorporate in the M-theory description of this background the effect of the O6-plane. The geometry that one gets is that of a  $D_N$  ALE singularity fibered over the base  $\mathbf{S}^2$ . This Calabi-Yau appears in [10] and M-theory on it gives a supersymmetric  $SO(2N)$  gauge theory. By analysing the resolution of this singularity reference [30] reproduced the exact prepotential in the Coulomb branch of the gauge theory from classical geometry<sup>11</sup>.

It is natural to ask for the M-theory description of wrapped D6-branes when the D6-branes wrap a collection of  $\mathbf{S}^2$ 's of a local K3 geometry. We will consider the well known ADE ALE spaces. The basic geometry is that of a collection of  $r$   $\mathbf{S}^2$ 's whose intersection form is  $I_{ab} = -C_{ab}$  where  $a, b = 1, \dots, r$ , which is (minus) the Cartan matrix of the corresponding ADE algebra of rank  $r$ . One can consider wrapping  $N_a$  D6-branes on the  $a$ -th  $\mathbf{S}^2$  for  $a = 1, \dots, r$ . In order to obtain a supersymmetric gauge theory on this collection of D6-branes, one must ensure that all the  $\mathbf{S}^2$ 's are holomorphic (supersymmetric) with respect to the same complex structure<sup>12</sup>. Then the gauge theory one obtains is a five

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<sup>11</sup> Both the  $SU(N)$  and  $SO(2N)$  pure gauge theories have a convex prepotential along the Coulomb branch and have a non-trivial fixed point of the renormalization group at the origin [27][10].

<sup>12</sup> For the  $A_r$  ALE spaces, for which an explicit metric is known [18], one can show that the  $r$  homology generators are holomorphic with respect to the same complex structure when the  $r$  vectors in  $\mathbf{R}^3$  – from which one constructs the  $\mathbf{S}^2$ 's by fiberling these vectors with the  $U(1)$  fiber – are collinear.

dimensional theory with eight real supercharges with gauge group

$$G = \prod_{a=1}^r SU(N_a) \quad (4.3)$$

and the following hypermultiplet content

$$\frac{1}{2} \oplus_{a \neq b} I_{ab}(N_a, \bar{N}_b). \quad (4.4)$$

This gauge theory can be easily derived by representing the wrapped D6-branes as fractional D4-branes probing the  $\mathbf{C}^2/\Gamma$  orbifold singularity, where  $\Gamma$  is the discrete subgroup of  $SU(2)$  whose representation theory can be associated with the extended ADE Dynkin diagram<sup>13</sup>.

The M-theory description of this Type IIA background is as follows. One has a fibration structure whose base is given by a chain of  $r$   $\mathbf{S}^2$ 's which intersect according to a particular ADE Dynkin diagram of rank  $r$ . Which ADE diagram appears is determined by which ADE ALE space is used in the Type IIA description. On the  $a$ -th  $\mathbf{S}^2$  in the base one has a  $A_{N_a-1}$  ALE singularity fibered over it so the total geometry is that of a chain of  $\mathbf{S}^2$ 's intersecting according to the ADE Dynkin diagram and on each  $\mathbf{S}^2$  there is an  $A$ -type singularity fibered over it. The type of singularity on a particular  $\mathbf{S}^2$  in the base is determined by the number of D6-branes that wrap that cycle in the Type IIA description. This geometry is a Calabi-Yau three-fold and M-theory on it results in the required quiver gauge theory<sup>14</sup> and reproduces the cubic prepotential along the Coulomb branch [10].

We conclude this section by identifying the Type IIA background which lifts to M-theory on a family of Calabi-Yau manifolds considered in [10]. In [10] it was shown that one can generalize the base geometry that we have just discussed such that the base  $\mathbf{S}^2$ 's intersect according to Dynkin diagram of the affine  $\hat{A}\hat{D}\hat{E}$  groups. The hypermultiplets are determined like in (4.4) but now  $C_{ab}$  is the Cartan matrix of  $\hat{A}\hat{D}\hat{E}$ . It is natural to try to identify the Type IIA geometry with  $SU(2)$  holonomy which lifts, in the presence of D6-branes, to this family of Calabi-Yau manifolds.

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<sup>13</sup>  $A_r = \mathbf{Z}_{r+1}$ ,  $D_r = \hat{\mathbf{D}}_{r-2}$ ;  $E_6, E_7$  and  $E_8$  correspond respectively to the  $\mathbf{Z}_2$  centrally extended tetrahedral, octohedral and isocahedral groups.

<sup>14</sup> The matter appears as usual from loci of enhanced symmetry in the base.

When the base  $\mathbf{S}^2$ 's intersect according to the  $\hat{A}_r$  Dynkin diagram, the corresponding Type IIA geometry is the deformation of the Type  $I_{r+1}$  fibre singularity of an elliptically fibered K3. This K3 is described by the Weierstrass form

$$y^2 = x^3 + a(z)x + b(z), \quad (4.5)$$

where  $(x, y)$  are affine coordinates on  $\mathbb{P}^2$  and  $z$  is the coordinate on the base of the elliptic fibration parametrizing a copy of  $\mathbf{C}$ . The Type  $I_{r+1}$  singularity appears when the discriminant of (4.5) has a zero of order  $r + 1$  at  $z = 0$  and both  $a(z)$  and  $b(z)$  don't have a zero at  $z = 0$ . The deformation of this singularity generates a chain of  $r + 1$   $\mathbf{S}^2$ 's which intersect according to the  $\hat{A}_r$  Dynkin diagram. Wrapping D6-branes on these two-cycles lifts to eleven dimensions to the Calabi-Yau three-fold we were looking for. One can find in a similar fashion the Type IIA realization when the  $\mathbf{S}^2$ 's in the base of the three-fold have a different affine intersection form. This corresponds to studying Type IIA with D6-branes on the deformation of certain fiber singularities – see e.g [22] – of an elliptically fibered K3.

## 5. From $SU(3)$ holonomy to $G_2$ holonomy via D6-Branes

For completeness, we briefly summarize an example of this lift which appeared recently in [8][9]. The Calabi-Yau three-fold they considered is  $T^*\mathbf{S}^3$ , the cotangent bundle of  $\mathbf{S}^3$ . This manifold appears as the deformation of the quadric on  $\mathbf{C}^4$

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = r. \quad (5.1)$$

In a similar fashion as in (4.1), one may show that for real  $r$  (5.1) describes  $T^*\mathbf{S}^3$ . One can now wrap  $N$  D6-branes over the  $\mathbf{S}^3$ , which is a Special Lagrangian submanifold. The branes completely fill the transverse  $\mathbf{R}^{1,3}$  Minkowski space and give rise a four dimensional  $SU(N)$  gauge theory with four supercharges.

In [8][9] the local description of this background was described as M-theory on a certain space with  $G_2$  holonomy. The D6-branes lead to an  $A_{N-1}$  ALE singularity which is fibered over the  $\mathbf{S}^3$ . This geometry can be understood as a  $\mathbf{Z}_N$  orbifold of an  $\mathbf{R}^4$  fibration over  $\mathbf{S}^3$ . Supersymmetry dictates that this space must have  $G_2$  holonomy. It is known that the spin bundle of  $\mathbf{S}^3$   $S(\mathbf{S}^3)$ , whose fibers are topologically  $\mathbf{R}^4$ , admits a metric with  $G_2$  holonomy [31]. Moreover, the  $G_2$  structure on this space contains an  $SU(2)^3$  symmetry

group. Therefore, modding out by any discrete subgroup of  $SU(2)^3$  will preserve the  $G_2$  structure. Indeed, it is easy to show that embedding  $\mathbf{Z}_N$  in one of the  $SU(2)$ 's acts on the fibers precisely as we expect from the lift of Type IIA D6-branes on  $T^*(\mathbf{S}^3)$ . Therefore, the Type IIA configuration is represented in eleven dimensions by a  $\mathbf{Z}_N$  orbifold of  $S(\mathbf{S}^3)$  which has a  $G_2$  structure.

Sinha and Vafa [12] considered the Type IIA orientifold background obtained by modding out by the following involution

$$\sigma : z_i \rightarrow \bar{z}_i \quad i = 1, \dots, 4. \quad (5.2)$$

This involution gives rise to an O6-plane wrapping the  $\mathbf{S}^3$ , which is left fixed by (5.2). The gauge theory on the D6-branes on top of this O6-plane is a four dimensional  $\mathcal{N} = 1$   $SO(2N)$  gauge theory.

From the general discussion in section 3 such background is described in eleven dimensions by a  $D_N$  ALE singularity fibered over  $\mathbf{S}^3$ . This is obtained by having the discrete group  $\hat{\mathbf{D}}_{N-2}$  act on the  $\mathbf{R}^4$  fibers. Since  $\hat{\mathbf{D}}_{N-2} \in SU(2)$ , one can mod out  $S(\mathbf{S}^3)$  by  $\hat{\mathbf{D}}_{N-2}$  by appropriately embedding  $\hat{\mathbf{D}}_{N-2}$  in one of the  $SU(2)$  symmetry groups of  $S(\mathbf{S}^3)$  such that the quotient space has a  $G_2$  structure and  $\mathbf{D}_{N-2}$  acts as required on the fibers.

Various aspects of this lift and generalizations to other three-folds have recently appeared in [12,13,14,15].

## 6. From $SU(3)$ holonomy to $SU(4)$ holonomy via D6-Branes

In order to accomplish this lift the D6-branes must be codimension one in the transverse  $\mathbf{R}^{1,3}$  Minkowski space. Therefore, the D6-branes must wrap a divisor in the Calabi-Yau three-fold. In this section we will provide a simple example of this lift.

Consider the non-compact Calabi-Yau geometry described by the  $\mathcal{O}(-3)$  bundle over  $\mathbb{P}^2$ . The string sigma model on this three-fold has a simple description in terms of the linear sigma model approach of [32]. We analyze the vacuum structure of a two dimensional  $\mathcal{N} = (2, 2)$   $U(1)$  gauge theory with four chiral superfields  $(z_1, z_2, z_3, z_4)$  which carry charges  $(1, 1, 1, -3)$  under the gauge group. In the presence of a Fayet-Iliopoulos term  $r$ , the solution of the D-term equation of this model is described by

$$M : \quad |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 = r. \quad (6.1)$$

The vacuum of the theory is  $M/U(1)$ , where the  $U(1)$  action is specified by the  $U(1)$  charges of  $z_i$ .

The phase  $r > 0$  describes an exceptional  $\mathbb{P}^2$  parametrized by coordinates  $z_1, z_2, z_3$  which due to (6.1) cannot vanish simultaneously. The coordinate  $z_4$  describes a complex line fibered over  $\mathbb{P}^2$ . The vacuum manifold  $M/U(1)$  can be identified<sup>15</sup> with the  $\mathcal{O}(-3)$  bundle over  $\mathbb{P}^2$ . The exceptional  $\mathbb{P}^2$  is a holomorphic cycle so branes wrapping it are supersymmetric.

We consider  $N$  D6-branes wrapping the  $\mathbb{P}^2$ . The field theory on the branes is a three-dimensional  $\mathcal{N} = 2$  pure  $SU(N)$  gauge theory<sup>16</sup>. Classically, this theory has a Coulomb branch parametrized by  $N - 1$  complex scalars which are formed from the real scalars in the vector multiplet together with the real scalars one gets by dualizing the photons one gets along the Coulomb branch. Semiclassically, Yang-Mills instantons generate a superpotential for the complex scalars [35] break supersymmetry.

The local M-theory description of this background is given by an  $A_{N-1}$  ALE singularity fibered over  $\mathbb{P}^2$  which is a Calabi-Yau four-fold. Analyzing the zero mode spectrum leads to a  $\mathcal{N} = 2$  three-dimensional pure  $SU(N)$  gauge theory<sup>17</sup>. Precisely the four-fold geometry we have encountered can be used to exactly reproduce the superpotential along the Coulomb branch of the gauge theory [36]. In the geometrical picture going to the Coulomb branch corresponds to resolving the singularities on the fiber which lead in the total geometry to a collection of divisors. The vevs of the complex scalars in the Coulomb branch correspond to the complexified blow up parameters. The superpotential then arises, when certain topological conditions [37] are satisfied, by wrapping Euclidean five-branes over the divisors. It was shown in [36] that the divisors one obtains by deforming the singularity we discussed reproduces the expected field theory superpotential of the three dimensional  $SU(N)$  gauge theory.

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<sup>15</sup> This Calabi-Yau is the crepant resolution of the  $\mathbf{C}^3/\mathbf{Z}_3$  orbifold singularity which appears in the linear sigma model approach in the phase when  $r < 0$ ; then  $z_4$  cannot vanish and the exceptional divisor is blown down. More details on the moduli space of this model can be found in [33].

<sup>16</sup> There is a subtlety in this example. Since  $\mathbb{P}^2$  is not a Spin manifold we must turn on a half-integral flux on the D-brane in order to avoid global anomalies [34]. We will return to this issue in the next section.

<sup>17</sup> If the complex surface  $Z$  over which the singular ALE sits has non-zero Betti numbers  $h^{1,0}(Z)$  and  $h^{2,0}(Z)$  one gets chiral multiplets [36]. Since  $h^{1,0}(\mathbb{P}^2) = h^{2,0}(\mathbb{P}^2) = 0$  we get pure gauge theory.

We can orientifold the Type IIA background by modding out by the following involution

$$\sigma : \begin{cases} z_4 & \rightarrow & -z_4 \\ x_3 & \rightarrow & -x_3 \end{cases} . \quad (6.2)$$

The action of  $\sigma$  on the three-fold (6.1) leaves the  $\mathbb{P}^2$  fixed and gives rise to a curved orientifold plane. Therefore, we can have  $N$  D6-branes on top of the O6-plane wrapping the  $\mathbb{P}^2$  inside the  $\mathcal{O}(-3)$  bundle over  $\mathbb{P}^2$ . Thus, the gauge theory on the D6-branes is a three dimensional  $\mathcal{N} = 2$  pure  $SO(2N)$  gauge theory. The local M-theory description of this Type IIA system is described by the Calabi-Yau four-fold given by a  $D_N$  ALE singularity fibered over  $\mathbb{P}^2$ . This geometry was used in [36] to reproduce the superpotential on the Coulomb branch of the gauge theory using five-brane instantons.

## 7. From $G_2$ holonomy to $\text{Spin}(7)$ holonomy via D6-Branes

In order to accomplish this lift the D6-branes must fill the transverse  $\mathbf{R}^{1,2}$  Minkowski space. We must then look for manifolds with  $G_2$  holonomy which have a coassociative four cycle. In order to avoid complications with tadpoles we will consider non-compact geometries.

In the literature [31], there are only three complete metrics on seven manifolds which admit a  $G_2$  structure<sup>18</sup>. Of these geometries, two have a coassociative four cycle<sup>19</sup>. They are metrics on the bundle of anti-self-dual two-forms over the four-manifold  $\mathbf{Z} \Sigma(\mathbf{Z})$ , where  $\mathbf{Z} = \mathbf{S}^4$  or  $\mathbf{Z} = \mathbb{P}^2$ . Topologically, the fibers are  $\mathbf{R}^3$  and  $\mathbf{Z}$  is a supersymmetric four-cycle. Therefore, wrapping a collection of D6-branes over  $\mathbf{Z}$  gives rise to a three-dimensional  $\mathcal{N} = 1$  gauge theory. Since the normal bundle of  $\mathbf{Z}$  is trivial<sup>20</sup> one obtains a pure  $SU(N)$  gauge theory. Theories with three-dimensional  $\mathcal{N} = 1$  supersymmetry do not have the familiar constraints of holomorphy of four dimensional  $\mathcal{N} = 1$  field theories, therefore one cannot make exact statements about their non-perturbative dynamics.

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<sup>18</sup> A  $G_2$  structure is a globally defined three-form  $\Phi$  which is covariantly constant, closed, co-closed and invariant under the group  $G_2$ .  $G_2$  is the subgroup of  $SO(7)$  which leaves the multiplication table of imaginary octonions invariant.

<sup>19</sup> The other metric is a  $G_2$  holonomy metric on the spin bundle over  $\mathbf{S}^3 S(\mathbf{S}^3)$  which appeared in the previous section. This geometry has an associative three-cycle.

<sup>20</sup> The normal bundle is the bundle of anti-self-dual two-forms. Since  $\dim(H_-^2(\mathbf{Z}) = 0)$ , there are no scalars.

We will start by identifying the M-theory description when the D6-branes wrap the  $\mathbf{S}^4$  in  $\Sigma(\mathbf{S}^4)$ . The M-theory geometry is given by the  $A_{N-1}$  ALE singularity fibered over  $\mathbf{S}^4$  or equivalently by an  $\mathbf{R}^4$  bundle over  $\mathbf{S}^4$  where the cyclic group  $\mathbf{Z}_N$  acts on the  $\mathbf{R}^4$  fibers. Supersymmetry requires that this space admit a Spin(7) structure<sup>21</sup>. Fortunately, we can identify this space with an orbifold of the known [31] eight-dimensional space which admits a complete metric with Spin(7) holonomy. This space is the spin bundle over  $\mathbf{S}^4$   $S(\mathbf{S}^4)$ , whose fibers are topologically  $\mathbf{R}^4$  and admits a Spin(7) structure with an  $SU(2) \times Sp(2)$  symmetry. One can now embed  $\mathbf{Z}_N$  on  $SU(2)$  while preserving the Spin(7) structure. Moreover, this embedding acts geometrically by natural action of  $\mathbf{Z}_N$  on the  $\mathbf{R}^4$  fibers. Therefore, we have identified the M-theory description of D6-branes wrapping the  $\mathbf{S}^4$  in  $\Sigma(\mathbf{S}^4)$  as an abelian orbifold of  $S(\mathbf{S}^4)$  which preserves the Spin(7) structure.

One can also consider the case when the D6-branes wrap  $\mathbb{P}^2$  in  $\Sigma(\mathbb{P}^2)$ . Since  $\mathbb{P}^2$  is not a spin manifold, the field strength  $F$  of the  $U(N)$  “gauge field” on the D6-branes<sup>22</sup> does not obey conventional Dirac quantization [34]. One has

$$\int_{\mathbb{P}^1} \frac{F}{2\pi} = n + \frac{1}{2}, \quad (7.1)$$

where  $n$  is an integer and  $\mathbb{P}^1$  generates  $H_2(\mathbb{P}^2, \mathbf{Z})$ . In particular, the flux cannot vanish. Despite this flux, the effective three-dimensional theory one obtains on the branes is also a pure  $SU(N)$  gauge theory.

The eleven dimensional description of this Type IIA background is somewhat subtle. Naively, one gets near the location of the D6-branes an  $A_{N-1}$  ALE singularity fibered over  $\mathbb{P}^2$ . Supersymmetry suggest, in analogy with the previous example, that it must be possible to put a metric on an  $\mathbf{R}^4$  bundle over  $\mathbb{P}^2$  which is complete and has Spin(7) holonomy. Moreover, such Spin(7) structure must have an  $SU(2)$  symmetry on which one can embed  $\mathbf{Z}_N$  in such a way that it acts on the  $\mathbf{R}^4$  fibers in the usual fashion. We have not found in the literature a metric such an  $\mathbf{R}^4$  bundle over  $\mathbb{P}^2$  with Spin(7) holonomy but duality suggest that it must exist.

Just like in previous sections it is possible to consider the M-theory description when there is an orientifold six-plane wrapping  $\mathbf{Z}$ . The appropriate involution is given by acting by  $-\mathbf{1}$  on the fibers such that  $\mathbf{Z}$  is fixed by the involution. One then gets a three-dimensional

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<sup>21</sup> A Spin(7) structure is a globally defined self-dual four-form  $\Omega$  which is covariantly constant, closed and invariant under the group Spin(7).

<sup>22</sup> More precisely  $F$  is not a connection on a vector bundle but rather a  $\text{Spin}^c$  structure.



$\mathcal{N} = 1$   $SO(2N)$  gauge theory on the D-branes. The corresponding M-theory geometry is given by a  $D_N$  ALE singularity fibered over  $\mathbf{Z}$  which admits a  $\text{Spin}(7)$  structure.

The analysis of lifts to M-theory of wrapped D6-branes on supersymmetric cycles which are not Spin raises an interesting question. As we have remarked, absence of global anomalies forces [34], even for a single wrapped D6-brane over  $\mathbb{P}^2$ , to turn on a half-integral flux (7.1). In the absence of such a flux one expects the M-theory description to be given by Taub-NUT space. In the presence of this flux, it is not clear what must be the appropriate modification to the M-theory solution. It would be interesting to identify<sup>23</sup> the M-theory origin of the constraint on Dirac quantization whenever a D6-brane wraps a manifold which is not spin but is  $\text{Spin}^c$ .

### Acknowledgements:

We would like to thank M. Atiyah, A. Brandhuber, R. Corrado, E. Diaconescu, J. Gauntlett, S. Gukov, L. Motl, H. Ooguri and E. Witten for very useful discussions. Part of this work was done during the M-theory workshop at the ITP. We would like to thank the organizers for providing a stimulating atmosphere. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949 and by the DOE under grant no. DE-FG03-92-ER 40701.

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<sup>23</sup> This constraint was derived [38] from an M-theory perspective when one considers instead D4-branes. Then such a condition can be derived from an eleven dimensional viewpoint by demanding consistency of the M-theory five-brane partition function.

## References

- [1] P.K. Townsend, “The eleven-dimensional supermembrane revisited”, Phys.Lett. **B350** (1995) 184.
- [2] E. Witten, “String Theory Dynamics In Various Dimensions”, Nucl.Phys. **B443** (1995) 85.
- [3] O.Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y.Oz, “Large N Field Theories, String Theory and Gravity”, Phys.Rept. **323** (2000) 183; M.J. Duff, “TASI Lectures on Branes, Black Holes and Anti-de Sitter Space”, hep-th/9912164.
- [4] A. Strominger, “Superstrings with Torsion”, Nucl.Phys. **B274** (1986) 253.
- [5] K. Becker and M. Becker, “M-Theory on Eight-Manifolds”, Nucl.Phys. **B477** (1996) 155.
- [6] M. Berger, “Sur les groupes d’holonomie homogène de variétés à connexion affines et des variétés riemanniennes”, Bull. Soc. Math. France **83** (1955) 279.
- [7] S. Sethi, C. Vafa and E. Witten, “Constraints on Low-Dimensional String Compactifications”, Nucl.Phys. **B480** (1996) 213, hep-th/9606122.
- [8] B.S. Acharya, “On Realising N=1 Super Yang-Mills in M theory”, hep-th/0011089.
- [9] M. Atiyah, J. Maldacena and C. Vafa, “An M-theory Flop as a Large N Duality”, hep-th/0011256.
- [10] S. Katz, P. Mayr and C. Vafa, “Mirror symmetry and Exact Solution of 4D N=2 Gauge Theories I”, Adv.Theor.Math.Phys. **1** (1998) 53.
- [11] C. Vafa, “Superstrings and Topological Strings at Large N”, hep-th/0008142.
- [12] S. Sinha and C.Vafa, “SO and Sp Chern-Simons at Large N”, hep-th/0012136.
- [13] B.S. Acharya, “Confining Strings from  $G_2$ -holonomy spacetimes”, hep-th/0101206.
- [14] B.S. Acharya and C. Vafa, “On Domain Walls of N=1 Supersymmetric Yang-Mills in Four Dimensions”, hep-th/0103011.
- [15] F. Cachazo, K. Intriligator and C. Vafa, “A Large N Duality via a Geometric Transition”, hep-th/0103067.
- [16] G.T. Horowitz and A. Strominger, “Black Strings and P-Branes”, Nucl.Phys. **B360** (1991) 197.
- [17] S. Hawking, “Gravitational Instantons”, Phys.Lett. **A60** (1977) 81; G.W. Gibbons and S.W. Hawking, “Gravitational Multi-Instantons”, Phys.Lett. **B78** (1978) 430.
- [18] T. Eguchi and A.J. Hanson, “Asymptotically Flat Self-Dual Solutions to Euclidean Gravity”, Phys.Lett. **74B** (1978) 249; T.Eguchi, P.B. Gilkey and A.J. Hanson, “Gravitation, Gauge Theories and Differential Geometry”, Phys.Rep. **66** (1980) 214.
- [19] N. Seiberg, “IR Dynamics on Branes and Space-Time Geometry”, Phys.Lett. **B384** (1996) 81.
- [20] N. Seiberg and E. Witten, “Gauge Dynamics And Compactification To Three Dimensions”, hep-th/9607163.

- [21] M.F. Atiyah and N.J. Hitchin, “Low-Energy Scattering of Nonabelian Monopoles”, Phys. Lett. **A107** (1985) 21-25; “Low-Energy Scattering of Nonabelian Magnetic Monopoles”, Phil.Trans.Roy.Soc.Lond. **A315** (1985) 459-469.
- [22] P.S. Aspinwall, “K3 Surfaces and String Duality”, hep-th/9611137.
- [23] K. Becker, M. Becker and A. Strominger, “Fivebranes, Membranes and Non-Perturbative String Theory”, Nucl.Phys. **B456** (1995) 130; H. Ooguri, Y. Oz and Z. Yin, “D-Branes on Calabi-Yau Spaces and Their Mirrors”, Nucl.Phys. **B477** (1996) 407; K. Becker, M. Becker, D.R. Morrison, H. Ooguri, Y. Oz and Z. Yin, “Supersymmetric Cycles in Exceptional Holonomy Manifolds and Calabi-Yau 4-Folds”, Nucl.Phys. **B480** (1996) 225.
- [24] M. Bershadsky, V. Sadov and C. Vafa, “D-Branes and Topological Field Theories”, Nucl.Phys. **B463** (1996) 420.
- [25] M. Mariño, R. Minasian, G. Moore and A. Strominger, “Nonlinear Instantons from Supersymmetric p-Branes”, JHEP **0001** (2000) 005.
- [26] J. Maldacena and C. Núñez, “Supergravity description of field theories on curved manifolds and a no go theorem”, hep-th/0007018; J. Maldacena and C. Nuñez, “Towards the large N limit of pure N=1 super Yang Mills”, Phys.Rev.Lett. **86** (2001) 588; B.S. Acharya, J.P. Gauntlett and N. Kim, “Fivebranes Wrapped On Associative Three-Cycles”, hep-th/0011190; J.P. Gauntlett, N. Kim and D. Waldram, “M-Fivebranes Wrapped on Supersymmetric Cycles”, hep-th/0012195; H. Niemer and Y. Oz, “Supergravity and D-branes Wrapping Supersymmetric 3-Cycles”, hep-th/0011288; C. Núñez, L.Y. Park, M. Schvellinger and T.A. Tran, “Supergravity duals of gauge theories from F(4) gauged supergravity in six dimensions”, hep-th/0103080.
- [27] N. Seiberg, “Five Dimensional SUSY Field Theories, Non-trivial Fixed Points and String Dynamics”, Phys.Lett. **B388** (1996) 753.
- [28] M.R. Douglas and G. Moore, “D-branes, Quivers, and ALE Instantons”, hep-th/9603167; C.V. Johnson and R.C. Myers, “Aspects of Type IIB Theory on ALE Spaces”, Phys.Rev. **D55** (1997) 6382.
- [29] J. Polchinski, “Tensors from K3 Orientifolds”, Phys. Rev. **D55** (1997) 6423; M.R. Douglas, “Enhanced Gauge Symmetry in M(atrix) Theory”, JHEP **9707** (1997) 004; D.E. Diaconescu, M.R. Douglas, and J. Gomis, “Fractional Branes and Wrapped Branes”, JHEP **9802** (1998) 013.
- [30] K. Intriligator, D.R. Morrison and N. Seiberg, “Five-Dimensional Supersymmetric Gauge Theories and Degenerations of Calabi-Yau Spaces”, Nucl.Phys. **B497** (1997) 56.
- [31] R. Bryant and S. Salomon, “On the construction of some complete metrics with exceptional holonomy”, Duke Math. J. **58** (1989) 829; G.W. Gibbons, D.N. Page and C.N. Pope, “Einstein Metrics on  $S^3, R^3$  and  $R^4$  Bundles”, Commun.Math.Phys. **127** (1990) 529.

- [32] E. Witten, "Phases of  $N = 2$  Theories In Two Dimensions", Nucl.Phys. **B403** (1993) 159.
- [33] D.E. Diaconescu and J. Gomis, "Fractional Branes and Boundary States in Orbifold Theories", JHEP **0010** (2000) 001.
- [34] D.S. Freed and E. Witten, "Anomalies in String Theory with D-Branes", hep-th/9907189.
- [35] I.A. Affleck, J.A. Harvey and E. Witten, "Instantons and (Super)Symmetry Breaking in (2+1)-Dimensions", Nucl.Phys. **B206** (1982) 413.
- [36] S. Katz and C. Vafa, "Geometric Engineering of N=1 Quantum Field Theories", Nucl.Phys. **B497** (1997) 196.
- [37] E. Witten, "Non-Perturbative Superpotentials In String Theory", Nucl.Phys. **B474** (1996) 343.
- [38] E. Witten, "Duality Relations Among Topological Effects In String Theory", JHEP **0005** (2000) 031.